Exam Calculus 2

28 January 2025, 18:15-20:15

university of groningen

The exam consists of 4 problems. You have 120 minutes to answer the questions. You can achieve 100 points which includes a bonus of 10 points. Calculators, books and notes are not permitted.

1. **[4+8+8=20 Points]**

   Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x,y) = \begin{cases} x+y & \text{if } x \cdot y \geq 0 \\ x-y & \text{if } x \cdot y < 0 \end{cases} .$$

   (a) Show that f is continuous at $(x,y) = (0,0)$.

   (b) Show from the definition of directional derivatives that for each unit vector $\boldsymbol{u} = (v,w) \in \mathbb{R}^2$, the directional derivative $D_{\boldsymbol{u}} f(0,0)$ exists.

   (c) Is f differentiable at $(x,y) = (0,0)$? Give a detailed justification of your answer based on the definition of differentiability.

2. **[5+10+10=25 Points]**

   Let $S \subset \mathbb{R}^3$ be the elliptic paraboloid given by the equation

$$x^2 + 2y^2 - 6x - z + 10 = 0$$

   which contains the point $(x_0, y_0, z_0) = (4, 1, 4)$.

   (a) Find the tangent plane at the point (x_0, y_0, z_0) using the fact that S is the level set of a suitable function $g : \mathbb{R}^3 \to \mathbb{R}$.

   (b) Use the Implicit Function Theorem to show that near the point (x_0, y_0, z_0), the surface S can be considered to be the graph of a function f of the variables y and z. Compute the partial derivatives f_y and f_z at (y_0, z_0) and show that the tangent plane found in part (a) coincides with the graph of the linearization of f at (y_0, z_0).

   (c) Use the method of Lagrange multipliers to find the point(s) in S closest to the x-axis.

— please turn over —

3. **[6+5+6+3=20 Points]**

For the constant $a \in \mathbb{R}$, consider the vector field $\mathbf{F}$ on $\mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) = (axy - z^3)\,\mathbf{i} + (a-2)x^2\,\mathbf{j} + (1-a)xz^2\,\mathbf{k}\,, \quad (x, y, z) \in \mathbb{R}^3\,,$$

(a) Let $A = (0, 0, 0)$ and $B = (1, 1, 1)$, and C be the straight line segment connecting A to B. Compute the line integral of $\mathbf{F}$ along C, i.e.

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{s}.$$

(b) Determine a in such a way that the vector field $\mathbf{F}$ is conservative.

(c) Determine for the value of a found in part (b) a potential function of $\mathbf{F}$.

(d) Show that the potential function in part (c) can be used to compute the value of the line integral in part (a) in the case where $\mathbf{F}$ is conservative by using the Fundamental Theorem of Line Integrals.

4. **[25 Points]**

Let S be the part of the paraboloid $z = 9 - x^2 - y^2$ contained in the cylinder of radius 3 centered at the z-axis. Suppose S is oriented by the upward pointing normal vector. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined as

$$\mathbf{F}(x, y, z) = (2z - y)\,\mathbf{i} + (x + z)\,\mathbf{j} + (3x - 2y)\,\mathbf{k}$$

for all $(x, y, z) \in \mathbb{R}^3$. For this example, very Stokes' Theorem by computing both sides of the equality

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{s},$$

where the orientation on the boundary ∂S is induced by the orientation on S.

Solutions

1. (a) We have $f(0,0) = 0$. For all $(x,y) \in \mathbb{R}^2$, it hols that

$$0 \le |f(x,y)| = |x \pm y| \le |x| + |y|$$

where in the last inequality we used the Triangle Inequality. From the Squeezing Theorem we hence get $f(x,y) \to 0$ for $(x,y) \to 0$ which agrees with $f(0,0)$. The function f is hence continues at $(x,y) = (0,0)$.

(b) Let $\boldsymbol{u} = (v,w) \in \mathbb{R}^2$ with $v^2 + w^2 = 1$. Then for $0 \ne h \in \mathbb{R}$,

$$\begin{aligned}
&\frac{f(hv,hw) - f(0,0)}{h} \\
&= \frac{1}{h}\begin{cases} hv + hw & \text{if } hv \cdot hw \ge 0 \\ hv - hw & \text{if } hv \cdot hw < 0 \end{cases} \\
&= \frac{1}{h}\begin{cases} hv + hw & \text{if } v \cdot w \ge 0 \\ hv - hw & \text{if } v \cdot w < 0 \end{cases} \\
&= \begin{cases} v + w & \text{if } v \cdot w \ge 0 \\ v - w & \text{if } v \cdot w < 0 \end{cases}
\end{aligned}$$

which does not depend on h. Hence the limit $h \to 0$ trivially exists and

$$D_{\boldsymbol{u}} f(0,0) = \lim_{h\to 0} \frac{f(hv,hw) - f(0,0)}{h} = \begin{cases} v + w & \text{if } v \cdot w \ge 0 \\ v - w & \text{if } v \cdot w < 0 \end{cases}.$$

(c) According to part (b) we have $f_x(0,0) = f_y(0,0) = 1$ (choose $\boldsymbol{u} = (v,w) = (1,0)$ or $\boldsymbol{u} = (v,w) = (0,1)$, respectively). So the linearization of f at $(x,y) = (0,0)$ is given by

$$L(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = x + y.$$

For the differentiability of f at $(0,0)$, it needs to hold that the limit of

$$\frac{f(x,y) - L(x,y)}{\|(x,y) - (0,0)\|}$$

exists and is 0 for $(x,y) \to (0,0)$. Filling in f and L we get

$$\begin{aligned}
\frac{f(x,y) - L(x,y)}{\|(x,y) - (0,0)\|} &= \frac{1}{(x^2+y^2)^{1/2}}\begin{cases} x + y - (x+y) & \text{if } x \cdot y \ge 0 \\ x - y - (x+y) & \text{if } x \cdot y < 0 \end{cases} \\
&= \frac{1}{(x^2+y^2)^{1/2}}\begin{cases} 0 & \text{if } x \cdot y \ge 0 \\ -2y & \text{if } x \cdot y < 0 \end{cases}
\end{aligned}$$

which does not have a limit for $(x,y) \to (0,0)$. This can be seen, e.g., when $x = y$, then the latter equals 0 whereas for $y = -x \ne 0$, the latter is ± 1 depending on the sign of x. The function f is hence not differentiable at $(x,y) = (0,0)$.

The function is in fact piece-wise linear. There are two candidates for the tangent plane to the graph of f at $(x,y,z) = (0,0,f(0,0)) = (0,0,0)$. Namely the planes $z = x + y$ and $z = x - y$. These planes are however not equal. The function f is hence not differentiable at $(x,y) = (0,0)$.

2. (a) Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as

$$g(x,y,z) = x^2 + 2y^2 - 6x - z + 10$$

for $(x,y,z) \in \mathbb{R}^3$. Then S is the zero-level set of g. For the gradient of g, we get

$$\nabla g(x,y,z) = (2x-6, 4y, -1)$$

which at $(x_0, y_0, z_0) = (4,1,4)$ is

$$\nabla g(4,1,4) = (2,4,-1).$$

The tangent plane of S at $(x_0,y_0,z_0) = (4,1,4)$ is hence given by the equation

$$\begin{aligned} & \nabla g(x_0,y_0,z_0) \cdot (x-x_0, y-y_0, z-z_0) = 0 \\ \Leftrightarrow & (2,4,-1) \cdot (x-4, y-1, z-4) = 0 \\ \Leftrightarrow & 2x + 4y - z = 8. \end{aligned}$$

(b) We have

$$\frac{\partial g}{\partial x}(x_0,y_0,z_0) = 2 \neq 0.$$

By the Implicit Function Theorem there exists a neighborhood $U \subset \mathbb{R}^2$ of $(y_0,z_0) \in \mathbb{R}^2$, a neighborhood $V \subset \mathbb{R}$ of $x_0 \in \mathbb{R}$ and a function $f : U \to \mathbb{R}$ with such that for $(y,z) \in U$ and $x \in V$,

$$g(x,y,z) = 0 \Leftrightarrow x = fy, z).$$

Moreover

$$f_y(x_0,z_0) = -\frac{g_y(x_0,y_0,z_0)}{g_x(x_0,y_0,z_0)} = -\frac{4}{2} = -2$$

and

$$f_z(x_0,z_0) = -\frac{g_z(x_0,y_0,z_0)}{g_x(x_0,y_0,z_0)} = -\frac{-1}{2} = \frac{1}{2}.$$

The linearization of f at (y_0,z_0) is

$$L(x,z) = f(y_0,z_0) + f_y(y_0,z_0)(y-y_0) + f_z(y_0,z_0)(z-z_0) = 4 - 2(y-1) + \frac{1}{2}(z-4).$$

The graph of L is

$$x = 4 - 2(y-1) + \frac{1}{2}(z-4) \Leftrightarrow 2x + 4y - z = 8.$$

which agrees with the equation for the tangent plane found in part (a).

3. (a) The line segment C from A to B has the parametrization $\mathbf{r}(t) = (t,t,t)$ with $t \in [0,1]$. The line integral is then given by

$$\begin{aligned} \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t &= \int_0^1 ((at^2 - t^3)\,\mathbf{i} + (a-2)t^2\,\mathbf{j} + (1-a)t^3\,\mathbf{k}) \cdot (\mathbf{i}+\mathbf{j}+\mathbf{k})\,\mathrm{d}t \\ &= \int_0^1 (at^2 - t^3 + (a-2)t^2 + (1-a)t^3)\,\mathrm{d}t \\ &= \int_0^1 ((2a-2)t^2 - at^3)\,\mathrm{d}t \\ &= \left[\frac{2a-2}{3}t^3 - \frac{a}{4}t^4\right]_{t=0}^{t=1} \\ &= \frac{2a-2}{3} - \frac{a}{4}. \end{aligned}$$

(b) For $\mathbf{F}$ to be conservative the curl of $\mathbf{F}$ has to vanish. We have

$$\begin{aligned}
\nabla \times \mathbf{F}(x,y,z) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} \\
&= \big(\partial_y(1-a)xz^2 - \partial_z(a-2)x^2\big)\,\mathbf{i} + \\
&\quad \big(\partial_z(axy - z^3) - \partial_x(1-a)xz^2\big)\,\mathbf{j} + \\
&\quad \big(\partial_x(a-2)x^2 - \partial_y(axy - z^3)\big)\,\mathbf{k} \\
&= 0\,\mathbf{i} + \big(-3z^2 - (1-a)z^2\big)\,\mathbf{j} + \big(2(a-2)x - ax\big)\,\mathbf{k}.
\end{aligned}$$

Equating this to zero gives $a = 4$.

(c) Let f denote the potential function. Then f satisfies the equations

$$f_x = 4xy - z^3, \tag{1}$$
$$f_y = 2x^2, \tag{2}$$
$$f_z = -3xz^2. \tag{3}$$

Integrating Eq. (1) with respect to x gives

$$f(x,y,z) = 2x^2y - xz^3 + g(y,z),$$

where $g(y,z)$ is a integration constant which can dependent on y and z. Differentiating with respect to y and using Eq. (2) yields

$$2x^2 + g_y(y,z) = 2x^2,$$

i.e., $g_y(y,z) = 0$. So g does not dependent on y and is hence of the form $g(y,z) = h(z)$ for some function $h : \mathbb{R} \to \mathbb{R}$. So $f(x,y,z) = 2x^2y - xz^3 + h(z)$. Differentiating with respect to z and using Eq. (3) yields

$$-3xz^2 + h'(z) = -3xz^2$$

which gives $h'(z) = 0$, i.e. h is constant. So the potential function is

$$f(x,y,z) = 2x^2y - xz^3 + c$$

with $c \in \mathbb{R}$.

(d) According to the Fundamental Theorem for Line Integrals the line integral is given by $f(B) - f(A) = f(1,1,1) - f(0,0,0) = (2-1) - 0 = 1$, where f is the potential function computed in part (c). This agrees with the result found in part (a) for $a = 4$.

4. We start by computing the flux on the left hand side of Stokes' equality. The curl of $\mathbf{F}$ is

$$\begin{aligned}
\nabla \times \mathbf{F}(x,y,z) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z - y & x + z & 3x - 2y \end{vmatrix} \\
&= (-2-1)\,\mathbf{i} - (3-2)\,\mathbf{j} + (1-(-1))\,\mathbf{k} \\
&= -3\,\mathbf{i} - 1\,\mathbf{j} + 2\,\mathbf{k}.
\end{aligned}$$

Let $D \subset \mathbb{R}^2$ be the disk of radius 3 centered at the origin. We can then parametrize S by $X : D \to \mathbb{R}^3$, $(x, y) \mapsto \mathbf{X}(x, y)$ with

$$\mathbf{X}(x, y) = (x, y, 9 - x^2 - y^2).$$

Then

$$\mathbf{X}_x(x, y) = (1, 0, -2x) \text{ and } \mathbf{X}_y(x, y) = (0, 1, -2y)$$

which gives

$$\mathbf{X}_x \times \mathbf{X}_y = (2x, 2y, 1)$$

which is a normal vector consistent with the given orientation of S. Hence

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} &= \iint_D (\nabla \times \mathbf{F})(\mathbf{X}) \cdot (\mathbf{X}_x \times \mathbf{X}_y) \mathrm{d}A \\
&= \iint_D (-3, -1, 2) \cdot (2x, 2y, 1) \mathrm{d}A \\
&= \iint_D (-6x - 3y + 2) \mathrm{d}A.
\end{aligned}$$

As by symmetry $\iint_D -6x \mathrm{d}A = \iint_D -3y \mathrm{d}A = 0$ we get that $\iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = 2$ times the are of a disk of radius 3 which gives $2\pi 3^2 = 18\pi$.

We now compute the line integral on the right hand side of Stokes' equality. The boundary ∂S is the circle of radius 3 in the (x, y)-plane centered at the origin and can be parametrized by

$$\mathbf{r}(t) = (3 \cos t, 3 \sin t, 0)$$

with $t \in [0, 2\pi]$. The tangent vector $\mathbf{r}'(t) = (-3 \sin t, 3 \cos t, 0)$ defines an orientation on ∂S that is consistent with the orientation induced by the orientation on S.

Hence

$$\oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{s} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, \mathrm{d}t.$$

Using

$$\mathbf{F}(\mathbf{r}(t)) = (-3 \sin t, 3 \cos t, 3 \cos t - 6 \sin t)$$

we get

$$\begin{aligned}
\oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{s} &= \int_0^{2\pi} (-3 \sin t, 3 \cos t, 3 \cos t - 6 \sin t) \cdot (-3 \sin t, 3 \cos t, 0) \, \mathrm{d}t \\
&= \int_0^{2\pi} (9 \sin^2 t + 9 \cos^2 t) \, \mathrm{d}t \\
&= \int_0^{2\pi} 9 \, \mathrm{d}t \\
&= 18\pi
\end{aligned}$$

which agrees with left hand side of Stokes' equality computed above.